

Computing the length of a Bézier path

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October 17, 2023

1 Introduction

Bézier paths, as used in practice, consist of multiple segments, each of which is either a straight-line segment, a quadratic Bézier curve segment, or a cubic Bézier curve segment. These curves can be represented by the following expressions:

- $\mathbf{p}(t) = \mathbf{p}_0 B_{0,1}(t) + \mathbf{p}_1 B_{1,1}(t)$ for linear segments,
- $\mathbf{p}(t) = \mathbf{p}_0 B_{0,2}(t) + \mathbf{p}_1 B_{1,2}(t) + \mathbf{p}_2 B_{2,2}(t)$ for quadratic segments,
- $\mathbf{p}(t) = \mathbf{p}_0 B_{0,3}(t) + \mathbf{p}_1 B_{1,3}(t) + \mathbf{p}_2 B_{2,3}(t) + \mathbf{p}_3 B_{3,3}(t)$ for cubic segments,

where \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 are control points, $B_{i,n}(t)$ is a Bernstein polynomial defined by the equation

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

with $\mathbf{p}(t)$ and the control points represented by vectors.

If the X and Y components of $\mathbf{p}(t)$ are $x(t)$ and $y(t)$ (with an obvious generalization to more dimensions), then the distance between two points on a segment is

$$d(t_1, t_2) = \int_{t=t_1}^{t_2} \sqrt{A(t)^2 + B(t)^2} dt$$

where

$$\begin{aligned} A(t) &= \frac{dx}{dt} \\ B(t) &= \frac{dy}{dt} \end{aligned}$$

These derivatives can be computed by using the relation,

$$\frac{d}{dt}B_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t))$$

with $B_{-1,n-1}(t) = B_{n,n-1}(t) = 0$. This relation can be used to differentiate x and y , representing the derivatives as polynomials using a Bézier basis and the derivatives can then be easily transformed to a monomial basis.

Because $A(t)$ and $B(t)$ are squared, the square root will always be positive unless $A(t)$ and $B(t)$ share a common root. In addition, $A(t)$ and $B(t)$ can be at most quadratic polynomials. When $A(t)$ and $B(t)$ share at least one root, we can write the integral as

$$d(t_1, t_2) = \int_{t=t_1}^{t_2} |P(t)|\sqrt{Q(t)} dt$$

where $P(t)$ and $Q(t)$ are polynomials and where the degree of Q is either 2 or 0.

2 Straight line segments

For linear line segments, $A(t)$ and $B(t)$ are constants and the integral is trivial: the length of a line segment is a constant multiplied by $t_2 - t_1$, with positive values indicating that the path parameter is increasing and negative values indicating that the path parameter is decreasing.

3 Quadratic line segments

For quadratic line segments, $A(t)$ and $B(t)$ are constants or first degree polynomials. If $A(t)$ and $B(t)$ have a common root, the integrand is the absolute value of a polynomial. Otherwise, the integral can be written as

$$d(t_1, t_2) = \int_{t=t_1}^{t_2} \sqrt{a + bt + ct^2} dt$$

That integral can be found in the CRC Standard Mathematical Tables, which states that it is

$$d(t_1, t_2) = \frac{(2ct + b)\sqrt{X}}{4c} \Big|_{t=t_1}^{t_2} + \frac{1}{2k} \int_{t=t_1}^{t_2} \frac{1}{\sqrt{X}} dt$$

where $q = 4ac - b^2$, $k = \frac{4c}{q}$ and $X = a + bt + ct^2$. In addition, $\int_{t=t_1}^{t_2} \frac{1}{\sqrt{X}} dt$ has different values for

- $c > 0$, in which case

$$\int \frac{dt}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log(\sqrt{X} + t\sqrt{c} + \frac{b}{2/\text{sqrt}c})$$

- $c > 0$ and $q < 0$, in which case

$$\int \frac{dt}{\sqrt{X}} = \frac{1}{\sqrt{c}} \sinh^{-1} \left(\frac{2ct + b}{\sqrt{4ac - b^2}} \right)$$

- $c < 0$, in which case

$$\int \frac{dt}{\sqrt{X}} = \frac{1}{\sqrt{-c}} \sin^{-1} \left(\frac{-2ct - b}{\sqrt{b^2 - 4ac}} \right)$$

For $d(t_1, t_2)$, there are a few special cases. If $q = 0$, the integral reduces to the integral of the absolute value of a first degree polynomial and that is a good approximation when q is very close to zero as well. Also, when $|c|$ is sufficiently small, one can change the integral to

$$\begin{aligned} d(t_1, t_2) &= \int_{t=t_1}^{t_2} \sqrt{a+bt} \sqrt{1 + \frac{ct^2}{a+bt}} dt \\ &\approx \int_{t=t_1}^{t_2} \left[\sqrt{a+bt} + \frac{ct^2}{2\sqrt{a+bt}} \right] dt \end{aligned}$$

and both terms in the approximate value have integrals in the CRC Standard Mathematical Tables:

$$\int \sqrt{a+bt} dt = \frac{2}{3b} \sqrt{(a+bt)^3} \tag{1}$$

$$\int \frac{t^2}{\sqrt{a+bt}} dt = \frac{3(8a^2 - 4abt + 3b^2t^2)}{15b^3} \sqrt{a+bt} \tag{2}$$

For integrals containing absolute values of polynomials, it is necessary to break the integral up into segments whose boundaries are the limits of integration and the roots, with no segment including a root internally. Finally, when the integral uses logarithms, it is possible for $a + bt + ct^2$ to be positive for $t \in [t_1, t_2]$ and the argument for the logarithm to be negative. For this case use $\log x = \log(-x) + i\pi$ for $x < 0$. While the integral is a complex number, the imaginary part is a constant and thus drops out of definite integrals.

4 Cubic line segments

For cubic line segments, $A(t)$ and $B(t)$ could be constants or first degree polynomials, but are typically second degree polynomials. If $A(t)$ and $B(t)$ have a common root, the integrand is the square root of a constant or a second degree polynomial $Q(t)$, multiplied by the absolute value of a polynomial $P(t)$. There are three cases:

1. if $A(t)$ and $B(t)$ share one common root, $P(t)$ is a first degree polynomial and $Q(t)$ is a second degree polynomial
2. if $A(t)$ and $B(t)$ share two roots or both have the same double root, then $P(t)$ is a second-degree polynomial and $Q(t)$ is a constant.
3. If there are no roots in common, $Q(t) = A(t)^2 + B(t)^2$ is a fourth degree polynomial with no real roots, and can be factored into two polynomials $Q_1(t)$ and $Q_2(t)$ that have no real roots and that have positive values for all values of t .

Cases 1 and 2 involve integrals that are listed in the CRC Standard Mathematical Tables, and use elementary functions. While each has several cases, these are similar to the ones outlined in Section 3. One additional integral is useful:

$$\int x\sqrt{X} dt = \frac{X\sqrt{X}}{3c} - \frac{b}{2c} \int \sqrt{X} dt$$

where $X = a + bt + ct^2$. This integral can be used for the case where $A(t)$ and $B(t)$ have one common root, in which case the integral is that for the absolute value of a first degree polynomial multiplied by the square root of a quadratic polynomial.

The third case, which requires integrating the square root of a quartic polynomial, uses elliptic integrals. If $Q_1(t)$ and $Q_2(t)$ are written as

$$\begin{aligned} Q_1(t) &= f_1 + g_1t + h_1t^2 \\ Q_2(t) &= f_2 + g_2t + h_2t^2, \end{aligned}$$

the integral can be found in paper by B. Carlson, *A Table of Elliptic Integrals: Two Quadratic Factors*.¹ Equation 2.45 on Page 169 of Carlson's paper,

¹B.C. Carlson, "A Table of Elliptic Integrals: Two Quadratic Factors," *Mathematics of Computation*, Volume 59, Number 199, July 1992, Pages 165–180. <<https://www.ams.org/journals/mcom/1992-59-199/S0025-5718-1992-1134720-4/S0025-5718-1992-1134720-4.pdf>>

labeled as [1, 1, 1, 1], contains this integral, and makes use of a number of variables Carlson defined before Equation 2.45. One obtains

$$\begin{aligned} \int_{t=y}^x \sqrt{Q_1(t)Q_2(t)} dt &= (\delta_{22}^2/h_2^2 - \delta_{11}^2/h_1^2)[\psi_0 H_0 + (\Delta_0 - \delta_{12}^2)R_f]/8 \\ &\quad - (3\psi_0^2 - 4h_1 h_2 \delta_{12}^2)(\Sigma + \delta_{12}^2 R_f)/(24h_1^2 h_2^2) \\ &\quad + [\Delta^2 R_f - \psi_0 A(1, 1, 1, 1)]/(12h_1 h_2) + E/(3h_1) \end{aligned}$$

where the quantities in this integral are defined below. In the following list of definitions, i has each of two values (1 and 2) and the functions R_F , R_J , R_C , and R_D are the Carlson symmetric forms of elliptic integrals:

$$\begin{aligned} H_0 &= \delta_{11}^2 \psi_0 [R_J(M^2, L_-^2, L_+^2, \Omega_0^2)/3 + R_C(a_0^2, b_0^2)/2]/h_1^2 \\ &\quad - X_0 R_C(T_0^2, V_0^2) \\ a_0^2 &= b_0^2 + \Lambda_0(\Lambda_+ - \Lambda_0)(\Lambda_0 - \Lambda_-) \\ b_0^2 &= (S^2/U^2 + \Lambda_0)\Omega_0^4 \\ V_0^2 &= \mu_0^2(S^2 + \Lambda_0 U^2) \\ T_0 &= \mu_0 S + 2h_1 h_2 \\ \mu_0 &= h_1/(\xi_1 \eta_1) \\ X_0 &= -(\xi_1' \xi_2 + \eta_1' \eta_2)/(x - y) \\ \Omega_0^2 &= M^2 + \Lambda_0 \\ \Lambda_0 &= \delta_{11}^2 h_2/h_1 \\ \psi_0 &= g_1 h_2 - g_2 h_1 \\ A(1, 1, 1, 1) &= \xi_1 \xi_2 - \eta_1 \eta_2 \\ S &= (\xi_1 \eta_1 \theta_2 + \xi_2 \eta_2 \theta_1)/(x - y)^2 \\ \Sigma &= G - \Delta_+ R_f + B \\ R_f &= R_F(M^2, L_-^2, L_+^2) \\ G &= 2\Delta \Delta_+ R_D(M^2, L_-^2, L_+^2)/3 + \Delta/(2U) \\ &\quad (\delta_{12}^2 \theta_1 - \delta_{11}^2 \theta_2)/(4\xi_1 \eta_1 U) \\ L_\pm^2 &= M^2 + \Delta_\pm \\ \Delta_\pm &= \delta_{12}^2 \pm \Delta \\ \Delta &= (\delta_{12}^4 - \delta_{11}^2 \delta_{22}^2)^{1/2} \\ \delta_{ij} &= (2f_i h_j + 2f_j h_i - g_i g_j)^{(1/2)} \\ M &= \zeta_1 \zeta_2/(x - y) \\ U &= (\xi_1 \eta_2 + \eta_1 \xi_2)/(x - y) \\ \zeta_i &= [(\xi_i + \eta_i)^2 - h_i(x - y)^2]^{(1/2)} \end{aligned}$$

$$\begin{aligned}
\theta_i &= \xi_i^2 + \eta_i^2 - h_i(x - y)^2 \\
E &= \xi_1' \xi_1^2 \xi_2 - \eta_1' \eta_1^2 \eta_2 \\
B &= \xi_1' \xi_2 - \eta_1' \eta_2 \\
\eta_1' &= (g_1 + 2h_1 y) / (2\eta_1) \\
\xi_1' &= (g_1 + 2h_1 x) / (2\xi_1) \\
\eta_i &= (f_i + g_i y + h_i y^2)^{1/2} \\
\xi_i &= (f_i + g_i x + h_i x^2)^{1/2}
\end{aligned}$$

Programs that define these variables should place them in the reverse order: the list uses the convention in which quantities are used before they are defined, the opposite of that used by Carlson.

An algorithm for factoring $Q(t)$ is described in the article *Factoring Quartic Polynomials: A Lost Art*.² The easiest way to proceed is to factor $Q(t)$ into three terms: a constant, and two quadratic polynomials whose t^2 coefficients are 1.0. There are two special cases.

1. the minimum value of at least one of the two factors is small (e.g., less than 1×10^{-4}). In this case, tests indicate that numerical accuracy is poor, and one should use some other method such as numeric integration.
2. the two factors are identical. In this case, the integrand is the absolute value of a quadratic polynomial. For this case, the integral given in Carlson's paper can fail: a Java implementation returned `Double.NaN`, most likely because of division by zero, which suggests that Carlson's integral may not be numerically accurate when the two factors are nearly, but not exactly identical.

When the two factors are nearly identical (for example, when the coefficients differ by less than 10^{-7}), one can write

$$\begin{aligned}
\sqrt{Q_1(t)Q_2(t)} &= Q_1(t) \sqrt{\frac{Q_2(t)}{Q_1(t)}} = Q_1(t) \sqrt{1 + \frac{Q_2(t)}{Q_1(t)} - 1} \\
&\approx Q_1(t) \left[1 + \frac{1}{2} \left(\frac{Q_2(t)}{Q_1(t)} - 1 \right) \right] = \frac{Q_1(t) + Q_2(t)}{2}
\end{aligned}$$

and the integral can be approximated by the integral of a polynomial.

²Gary Brookfield, "Factoring Quartic Polynomials: A Lost Art", *Factoring Quartic Polynomials: A Lost Art*/A_i", *Mathematics Magazine*, Vol. 80, No. 1, February 2007, Pages 67–70, <https://www.maa.org/sites/default/files/Brookfield2007-103574.pdf>